Q-1 Solve the following integral equation

$$u(x) = 1 + \lambda \int_0^1 (x+t)\, u(t)\, dt$$

by the method of successive approximation upto third order.

Soln — Here the given integral equation is

$$u(x) = 1 + \lambda \int_0^1 (x+t)\, u(t)\, dt \quad \text{——— (1)}$$

Let us assume

$$u_0(x) = 1 \quad \text{——— (2)}$$

then the nth approximation is given by

$$u_n(x) = 1 + \lambda \int_0^1 (x+t)\, u_{n-1}(t)\, dt \quad \text{——— (3)}$$

$$\Rightarrow u_1(x) = 1 + \lambda \int_0^1 (x+t)\, u_0(t)\, dt$$

$$= 1 + \lambda \int_0^1 (x+t) \cdot 1 \cdot dt$$

$$= 1 + \lambda \left[xt + \frac{1}{2} t^2 \right]_0^1$$

$$= 1 + \lambda \left(x + \frac{1}{2} \right)$$

Also from equation (3), we get

$$u_2(x) = 1 + \lambda \int_0^1 (x+1)\, u_1(t)\, dt$$

$$= 1 + \lambda \int_0^1 (x+t) \left\{ 1 + \lambda \left(t + \frac{1}{2} \right) \right\} dt$$

$$= 1 + \lambda \int_0^1 \left[x\left(1+\frac{\lambda}{2}\right) + t\left[t + \frac{\lambda}{2} + \lambda x\right] + \lambda t^2 \right] dt$$

$$= 1 + \lambda \left[x\left(1+\frac{\lambda}{2}\right)t + \frac{t^2}{2}\left(1+\frac{\lambda}{2}+\lambda x\right) + \frac{\lambda t^3}{3} \right]_0^1$$

$$= 1 + \lambda\left(x+\frac{1}{2}\right) + \lambda^2\left[x + \frac{7}{12}\right] \quad \text{--- } (4)$$

Similarly,

$$u_3(x) = 1 + \lambda \int_0^1 (x+t)\, u_2(t)\, dt$$

$$= 1 + \lambda \int_0^1 (x+t) \left\{ 1 + \lambda\left(t+\frac{1}{2}\right) + \lambda^2\left(t+\frac{7}{12}\right) \right\} dt$$

$$= 1 + \lambda \int_0^1 (x+t) \left\{ \left(1+\frac{\lambda}{2}+\frac{7\lambda^2}{12}\right) + \lambda t(1+\lambda) \right\} dt$$

$$= 1 + \lambda \left[\left\{ x\left(1+\frac{\lambda}{2}+\frac{7\lambda^2}{12}\right)t + \frac{t^2}{2}\left(1+\frac{\lambda}{2}+\frac{7\lambda^2}{12} + x\lambda + x\lambda^2\right) + \frac{1}{3}\lambda t^3(1+\lambda) \right\} \right]_0^1$$

$$= 1 + \lambda x\left(1+\frac{\lambda}{2}+\frac{7\lambda^2}{12}\right) + \frac{\lambda}{2}\left(1+\frac{\lambda}{2}+\frac{7\lambda^2}{12} + \lambda x + x\lambda^2\right) + \frac{1}{3}\lambda^2(1+\lambda)$$

$$= 1 + \lambda\left(x+\frac{1}{2}\right) + \lambda^2\left(x+\frac{7}{12}\right) + \lambda^3\left(\frac{13}{12}x + \frac{5}{8}\right)$$